Solving Difference Equations and Inverse Z Transforms

Using Z Transforms To Solve Difference Equations

For LTI systems, described by linear constant coefficient difference equations

$$y[k]+a_1 y[k-1]+\ldots+a_n y[k-n]$$
$$=b_0 x[k]+b_1 x[k-1]+\ldots+b_m x[k-m]$$

current and past outputs

current and past inputs

Using Z Transforms To Solve Difference Equations

$$Y(z)[1 + a_1 z^{-1} + ... + a_n z^{-n}] = X(z)[b_0 + b_1 z^{-1} + ... + b_m z^{-m}]$$

Solve for the **Transfer Function H(z)** by dividing:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{[b_0 + b_1 z^{-1} + ... + b_m z^{-m}]}{[1 + a_1 z^{-1} + ... + a_n z^{-n}]}$$

$$= \frac{[b_0 z^n + b_1 z^{n-1} + ... + b_m z^{n-m}]}{[z^n + a_1 z^{n-1} + ... + a_n]}$$

Poles and Zeros

- Poles of H(z): roots of denominator polynomial
- Zeros of H(z): roots of numerator polynomial

  note: find these after canceling any common factors—and do this for polynomials in z (not z^{-1})

Using Z Transforms To Solve Difference Equations

- Find the output of an LTI system in the Z domain, *Y(z),* by multiplying the z-transform of the input, *X(z)* with *H(z)* = the Z transform of the impulse response
- Then you can use the Inverse Z Transform to get the output signal *y[k]* from its Z transform, *Y(z)*

Ex. Given a difference equation,

$$y[n]-.3y[n-1]=x[n]$$

find the z-transform of the equation and then find the response $Y(z)$ of the system to an input $x[n]=(.6)^n u[n]$.

First step—take z Transforms of both sides of the equation.
Since x[n] is given, we can use the z-transform tables to substitute for X(z).

$$Y[z]-0.3z^{-1}Y[z]=\frac{z}{z-.6}$$

X(z)

$$Y(z)[1-0.3z^{-1}]=\frac{z}{z-.6}$$

Factor out Y(z) –since we will want to inverse Z-transform it to get y[n]

Ex. Given a difference equation,

$$y[n]-.3y[n-1]=x[n]$$

find the z-transform of the equation and then find the response $Y(z)$ of the system to an input $x[n]=(.6)^n u[n]$.

$$Y[z]-0.3z^{-1}Y[z]=\frac{z}{z-.6}$$

$$Y(z)[1-0.3z^{-1}]=\frac{z}{z-.6}$$

$$Y(z)=\left(\frac{z}{z-.06}\right)\left(\frac{z}{z-0.3}\right)$$

Now put everything in terms of z, rather than having z^{-1} terms—and solve for Y(z)

What if you wanted to find the response in the time domain?

$\Rightarrow$ We can use **Partial Fraction Expansion** to invert the z-transform.

Similar to what you saw for Laplace Transforms,

$$Y(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\frac{N(z)}{D(z)}=\sum_{k=1}^{N}\frac{r_k z}{z-p_k}$$

$$p_k = \text{pole} \quad r_k = \text{residue}$$

where

$$r_k = [\frac{Y(z)}{z}(z-p_k)]\,|_{z=p_k}$$

For Distinct (non-repeated) roots

Then use tables to invert the z-transform, e.g.

$$a^n u[n] \leftrightarrow \frac{z}{z-a}$$

Properties of z-Transform

Property	Sequence	z-Transform	ROC
	$g[n]$	$G(z)$	R_g
	$h[n]$	$H(z)$	R_h
Linearity	$\alpha g[n]+\beta h[n]$	$\alpha G(z)+\beta H(z)$	Include $R_g \cap R_h$
Time-shifting	$g[n-k]$	$z^{-k}G(z)$	R_g except possibly the point $z=0$ or ∞
Multiplication by exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	αR_g
Convolution	$g[n]*h[n]$	$G(z)H(z)$	Include $R_g \cap R_h$
Time reversal	$g[-n]$	$G(1/z)$	$1/R_g$
Differentiation of $G(z)$	$ng[n]$	$-z\dfrac{dG(z)}{dz}$	R_g except possibly the point $z=0$ or ∞

Commonly used z-Transform pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n \mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$n\alpha^n \mu[n]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(n+1)\, \alpha^n \mu[n]$	$\dfrac{1}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n \cos \omega_o n)\, \mu[n]$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$
$(r^n \sin \omega_o n)\, \mu[n]$	$\dfrac{1-(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$

Returning to our example

$$y[n] - .3y[n-1] = x[n]$$

find the z-transform of the equation and then find the response $Y(z)$ of the system to an input $x[n] = (.6)^n u[n]$.

$$Y[z] - 0.3z^{-1}Y[z] = \frac{z}{z-.6}$$

$$Y(z)[1 - 0.3z^{-1}] = \frac{z}{z-.6}$$

$$Y(z) = \left(\frac{z}{z-.6}\right)\left(\frac{z}{z-0.3}\right)$$

$$\frac{Y(z)}{z} = \left(\frac{2}{z-.6}\right) + \left(\frac{-1}{z-0.3}\right)$$

Ex. Given a difference equation,

$$y[n] - .3y[n-1] = x[n]$$

find the z-transform of the equation and then find the response $Y(z)$ of the system to an input $x[n] = (.6)^n u[n]$.

$$Y[z] - 0.3z^{-1}Y[z] = \frac{z}{z-.6}$$

$$Y(z)[1 - 0.3z^{-1}] = \frac{z}{z-.6}$$

$$Y(z) = \left(\frac{z}{z-.6}\right)\left(\frac{z}{z-0.3}\right)$$

$$\frac{Y(z)}{z} = \left(\frac{2}{z-.6}\right) + \left(\frac{-1}{z-0.3}\right)$$

$$Y(z) = \left(\frac{2z}{z-.6}\right) + \left(\frac{-z}{z-0.3}\right)$$

↑ ↑

Now move the "saved" z back into the expression, so that these terms match things in the tables

Ex. Given a difference equation,

$$y[n] - .3y[n-1] = x[n]$$

find the z-transform of the equation and then find the response $Y(z)$ of the system to an input $x[n] = (.6)^n u[n]$.

$$Y[z] - 0.3z^{-1}Y[z] = \frac{z}{z-.6}$$

$$Y(z)[1 - 0.3z^{-1}] = \frac{z}{z-.6}$$

$$Y(z) = \left(\frac{z}{z-.6}\right)\left(\frac{z}{z-0.3}\right)$$

$$\frac{Y(z)}{z} = \left(\frac{2}{z-.6}\right) + \left(\frac{-1}{z-0.3}\right)$$

Finally, use tables to get *y[n]*

$$Y(z) = \left(\frac{2z}{z-.6}\right) + \left(\frac{-z}{z-0.3}\right)$$

$$y[n] = 2(.6)^n u[n] - (.3)^n u[n]$$

One-sided Z Transform

- Key property—useful when considering initial conditions of difference equations:

$$Z(x[k+1]) = \sum_{n=0}^{\infty} x[n+1]z^{-n}$$

$$= \left(\sum_{n=0}^{\infty} x[n]z^{-(n-1)}\right) - x[0]z$$

$$= z\sum_{n=0}^{\infty} x[n]z^{-n} - x[0]z$$

$$= zX(z) - zx[0]$$

Example: for output y and input u

$y[k+1]-0.8y[k]=x[k]$ with initial condition $y[0]=2$

and unit step input $x[k]=u[k]$

Take the one-sided Z transform of equation and input to get

$$[zY[z]-zy[0]]-0.8Y[z]=\frac{z}{z-1}$$

Solve for $Y(z)$ since we eventually want to get closed form solution $y[k]$ for the difference equation.

$$Y[z]=y[0]\frac{z}{z-0.8}+\frac{z}{(z-1)(z-0.8)}$$

$$Y[z]=y[0]\frac{z}{z-0.8}+\frac{z}{(z-1)(z-0.8)}$$

Need to partial fraction the second term:

$$\frac{1 \text{ (not } z)}{(z-0.8)(z-1)}=\frac{k_1}{z-0.8}+\frac{k_2}{z-1}$$

$$k_1=\left.\frac{1}{z-1}\right|_{z=0.8}=\frac{1}{-0.2}=-5$$

$$k_2=\left.\frac{1}{z-0.8}\right|_{z=1}=\frac{1}{0.2}=5$$

So

$$\frac{z}{(z-0.8)(z-1)}=\frac{k_1 z}{z-0.8}+\frac{k_2 z}{z-1}$$

$$=\frac{-5z}{z-0.8}+\frac{5z}{z-1}$$

Hence

$$Y[z]=y[0]\frac{z}{z-0.8}+\frac{-5z}{(z-0.8)}+\frac{5z}{z-1}$$

$$Y[z] = y[0]\frac{z}{z-0.8} + \frac{-5z}{(z-0.8)} + \frac{5z}{z-1}$$

Inverse Z transforms give

$$y[k] = \{[y[0]-5](0.8)^k + 5\}u[k]$$

So with y[0] = 2,
get solution:

$$\boxed{y[k] = \{-3(0.8)^k + 5\}u[k]}$$

Ex. Find the Inverse z-Transform of

$$X(z) = \frac{2z^2 - 5z}{(z-2)(z-3)},$$

We begin by dividing out one "**z**" to protect it for later use in with out inverse **z** transform table

$$\frac{X(z)}{z} = \frac{2z-5}{(z-2)(z-3)}$$

Ex. Find the Inverse z-Transform of

$$X(z)=\frac{2z^2-5z}{(z-2)(z-3)}$$

Expand:

$$\frac{X(z)}{z}=\frac{2z-5}{(z-2)(z-3)}=\frac{\frac{-1}{-1}}{z-2}+\frac{\frac{1}{1}}{z-3}$$

Next, bring back the "**z**" so that it matches the form in the table

$$X(z)=\frac{z}{z-2}+\frac{z}{z-3}$$

Ex. Find the Inverse z-Transform of

$$X(z)=\frac{2z^2-5z}{(z-2)(z-3)},|z|>3$$

Expand:

$$\frac{X(z)}{z}=\frac{2z-5}{(z-2)(z-3)}=\frac{\frac{-1}{-1}}{z-2}+\frac{\frac{1}{1}}{z-3}$$

$$X(z)=\frac{z}{z-2}+\frac{z}{z-3}$$

$$\therefore x[n]=(2)^n u[n]+(3)^n u[n]$$

Right-sided

Ex. Find the Inverse z-Transform of

$$X(z)=\frac{2z^2-5z}{(z-2)(z-3)}, |z|>3$$

$$X(z)=\frac{z}{z-2}+\frac{z}{z-3}$$

$$\therefore x[n]=(2)^n u[n]+(3)^n u[n]$$

Ex. Given $h[n]=a^n u[n]$ $(|a|<1)$ and $x[n]=u[n]$, find $y[n]=x[n]*h[n]$.

$$H(z)=\frac{z}{z-a} \qquad X(z)=\frac{z}{z-1}$$

$$\frac{Y(z)}{z}=\frac{z}{(z-a)(z-1)}=\frac{\frac{a}{a-1}}{z-a}+\frac{\frac{1}{1-a}}{z-1}$$

$$y[n]=\frac{a}{a-1}a^n u[n]+\frac{1}{1-a}u[n]$$

$$=\frac{1}{1-a}(1-a^{n+1})u[n]$$